Proposal for the Hilbert space structure for unconstrained vacuum general relativity

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Abstract

There is a new set of variables, which from our analysis appear naturally adapted to a Hilbert space description on the reduced phase space for vacuum GR. In this paper we present the Hilbert space, which features a basis of coherent-like states labelled by the algebraic classification of the corresponding spacetime. These wavefunctions, which satisfy the semiclassical-quantum correspondence, correspond to the solution of the quantum Hamiltonian constraint on the space of gauge-invariant, diffeomorphism invariant states in these variables, and are free of field-theoretical singularities.

1 Introduction

Metric general relativity is a totally constrained system whose action can be written as a canonical one-form minus a linear combination of first class constraints smeared by auxilliary fields

$$S_{EH}[g] = \int dt \int_{\Sigma} d^3x \big(\pi^{ij}\dot{h}_{ij} - NH - N^i H_i\big), \tag{1}$$

where the phase space variables are the 3-metric on a three dimensional spatial hypersurface Σ and its conjugate momentum, given by (h_{ij}, π^{ij}). In equation (1) H_i is the diffeomorphism constraint, given by

$$H_i = D_j \pi^j_i \sim 0, \tag{2}$$

where N^i is the shift vector and D_i is the three dimensional covariant derivative with respect to Σ. H is the Hamiltonian constraint, given by

$$H = G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{h}^{(3)}R \sim 0. \tag{3}$$

N is the lapse function and G_{ijkl} is the metric on superspace, given by

$$G_{ijkl} = \frac{1}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}). \tag{4}$$

Equations (2) and (3) signify the presence of 8 unphysical degrees of freedom, which if eliminated would reduce the phase space variables (h_{ij}, π^{ij}) to the 4 physical degrees of freedom needed to describe GR.[1] However, the constraint (3) remains unsolved in the full theory due to its nonpolynomial structure in the basic variables.

The constraints can be rewritten in terms of new variables $(A^a_i, \widetilde{\sigma}^i_a)$ attributed to Abhay Ashtekar, by canonical transformation from the ADM triad description of gravity [1],[2],[3]. In the Ashtekar variables the constraints in smeared form are given by

$$H_i[N^i] = \int_{\Sigma} d^3x N^i \epsilon_{ijk} \widetilde{\sigma}^j_a B^k_a \tag{5}$$

[1]The reduced phase space description would entail factoring out the gauge orbits generated by the constraints, on the constraint surface.

for the diffeomorphism constraint and

$$H[\underline{N}]=\int_{\Sigma}d^3x\underline{N}\Big(\frac{\Lambda}{6}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c+\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_bB^k_c\Big) \tag{6}$$

for the Hamiltonian constraint with cosmological constant Λ, where $\underline{N}=N/\sqrt{\det\widetilde{\sigma}}$ is the lapse density function and the shift vector N^i takes on the same meaning as in (1).[2] The Gauss' law constraint is given by

$$G_a[A^a_0]=\int_{\Sigma}d^3xA^a_0D_i\widetilde{\sigma}^i_a \tag{7}$$

which signifies invariance under $SU(2)_-$ rotations. A^a_0 is the $SU(2)$ rotation angle with covariant derivative $D_i\equiv(D_i)_{ab}=\delta_{ab}\partial_i+f_{abc}A^c_i$ and structure constants f_{abc}.

Equations (5) and (6) can be regarded as the direct analogues of (2) and (3), with the latter being simplified due to the polynomial nature of the constraints. However, the presence of an additional constraint (7) entails the existence of additional degrees of freedom which are unphysical. This may in some ways be seen as a tradeoff for the simplifications in the constraints if it is more difficult to eliminate these degrees of freedom in congruity with the remaining constraints.[3] Therefore, our first order of business will be to eliminate the superfluous degrees of freedom in general relativity, which entails the introduction of a new set of variables.

2 Transformation and solution to the constraints in Soo/CDJ variables

We would like to preserve the simplicity of the original Ashtekar variables, while at the same time reducing general relativity to its basic physical degrees of freedom. First make the substitution

$$\widetilde{\sigma}^i_a=\Psi_{ae}B^i_e \tag{8}$$

[2]The auxilliary variables $N^i(x)$ and $A^a_i(x)$ can be thought of as the parameters for spatial diffeomorphisms and $SU(2)_-$ gauge transformations within Σ, and the auxilliary variable $\underline{N}$ can be thought of as the parameter for deformations normal to Σ.

[3]In loop quantum gravity, the spin network states solve the Gauss' law constraint by construction. However, the Hamiltonian constraint remains unsolved in the full theory to the present author's knowledge, which leaves open the issue of dynamics. This notwithstanding, the loop approach has led to significant physical insights at the kinematical level of the phase space.

in order to eliminate the densitized triad $\widetilde{\sigma}^i_a$ as a basic variable. Equation (8), known as the CDJ Ansatz, for the purposes of this paper will be regarded as a mathematical identity defining vacuum GR for nondegenerate B^i_a and nondegenerate Ψ_{ae}. The CDJ matrix can further be parametrized by its symmetric and its antisymmetric parts

$$\Psi_{ae}=\epsilon_{aed}\psi_d+\lambda_{ae}=\epsilon_{aed}\psi_d+\lambda_f O_{fa}O_{fe}, \tag{9}$$

where $O_{ae}=(e^{\theta\cdot T})_{ae}$ is a complex orthogonal matrix parametrized by three complex angles $\vec{\theta}\equiv(\theta^1,\theta^2,\theta^3)$ and $T\equiv(T_1,T_2,T_3)$ are the generators of the $so(3,C)$ algebra in the adjoint representation. Hence, the symmetric part λ_{ae} of the CDJ matrix with eigenvalues $\lambda_f\equiv(\lambda_1,\lambda_2,\lambda_2)$ can be seen as the result of a Lorentz transformation

$$\lambda_{ae}=(e^{\theta\cdot T})_{af}\lambda_f(e^{-\theta\cdot T})_{fe} \tag{10}$$

from the frame of eigenvalues, where $\vec{\theta}=0$, into an arbitrary frame parametrized by $\vec{\theta}\neq 0$. Note that $Im[\vec{\theta}]$ and $Re[\vec{\theta}]$ correspond to rotations and boosts, respectively, in $SO(3,C)$ language.

Let us now revisit the constraints in the language of the CDJ matrix with a view toward extracting the essential degrees of freedom at the classical level. From (5), the smeared diffeomorphism constraint is given by

$$H_i[N^i]=\int_\Sigma d^3x\epsilon_{ijk}N^iB^j_aB^k_e\Psi_{ae}. \tag{11}$$

For $\det B\neq 0$, the inverse of the Ashtekar magnetic field $(B^{-1})^a_i$ exists and (11) can be written in unsmeared form as

$$(\det B)(B^{-1})^d_i\psi_d=0\ \forall x\in\Sigma \tag{12}$$

where we have used (9). The smeared Hamiltonian constraint (6) can be written in the form

$$H[N]=\int_\Sigma d^3xN\sqrt{\frac{\det B}{\det\Psi}}\Big(Var\Psi+\Lambda\det\Psi\Big) \tag{13}$$

where $Var\Psi=(\text{tr}\Psi)^2-\text{tr}\Psi^2$. While (13) is nonpolynomial in these variables, we will see that it is convenient to extract the polynomial part when quantizing the theory. Since we are restricting to nondegenerate configurations $\det B\neq 0$ and $\det\Psi\neq 0$, we can then focus on the part

$$Var\Psi + \Lambda \det\Psi = 0 \ \forall x \in \Sigma \tag{14}$$

in solving the constraint. We now compute the ingredients of the Hamiltonian constraint using the parametrization (9). For the determinant we obtain

$$\det\Psi = \frac{1}{6}\epsilon_{abc}\epsilon_{efg}(\lambda_{ae} + \epsilon_{aed_1}\psi_{d_1})(\lambda_{bf} + \epsilon_{bfd_2}\psi_{d_2})(\lambda_{cg} + \epsilon_{cgd_3}\psi_{d_3})$$
$$= \det(\lambda_{ae}) + \det(\epsilon_{aed}\lambda_d) + \frac{1}{2}\epsilon_{abc}\epsilon_{efg}\Big(\epsilon_{cgd}\lambda_{ae}\lambda_{bf}\psi_d + \epsilon_{bfd}\epsilon_{cgd'}\lambda_{ae}\psi_d\psi_{d'}\Big). \tag{15}$$

Using the fact that the determinant of an antisymmetric matrix of odd rank vanishes in combination with the annihilation of antisymmetric on symmetric indices, we end up with

$$\det\Psi = \det\lambda + \frac{1}{2}(\epsilon_{abc}\epsilon_{fbd})(\epsilon_{efg}\epsilon_{ed'g})\psi_d\psi_{d'}\lambda_{ae}$$
$$= \det\lambda + \frac{1}{2}\lambda_{ae}(\delta_{af}\delta_{cd} - \delta_{ad}\delta_{cf})(\delta_{ec}\delta_{fd'} - \delta_{ed'}\delta_{fc})\psi_d\psi_{d'}$$
$$= \det\lambda + \lambda_{ae}\psi_a\psi_e \tag{16}$$

where we have made use of epsilon symbol identities. Likewise, we compute the variance

$$Var\Psi = (\mathrm{tr}\lambda)^2 - (\lambda_{ae} + \epsilon_{aed}\psi_d)(\lambda_{ea} - \epsilon_{aed'}\psi_{d'}) = Var\lambda - 2\delta_{ae}\psi_a\psi_e. \tag{17}$$

The polynomial part of the Hamiltonian constraint (13), which we will from now on refer to as the Hamiltonian constraint, then is given by

$$H = Var\lambda + \Lambda\det\lambda + (\Lambda\lambda_{ae} - 2\delta_{ae})\psi_a\psi_e = 0 \ \forall x \in \Sigma. \tag{18}$$

Upon substitution of the parametrization (10), then (18) reduces to

$$H = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{\Lambda}{2} + \big(\lambda_f O_{fa}O_{fe} - 2\delta_{ae}\big)\psi_a\psi_e = 0. \tag{19}$$

Note that the complex orthogonal matrix O_{ae} has cancelled out from the first two terms of the Hamiltonian constraint, which depend only on the invariants of λ_{ae}. The last term, quadratic in ψ_d, depends explicitly on O_{ae} which in turn depends on the $SO(3,C)$ frame through $\vec{\theta}$.

We would rather like to interpret the Hamiltonian constraint as being independent of the $SO(3,C)$ frame and consider the angles $\vec{\theta}$ as not being

independent physical degrees of freedom. The most direct way to do this is to recognize that on the space of solutions to the diffeomorphism constraint (12), we must have that $\psi_d = 0$. For $\psi_d = 0$ the last two terms of (19) vanish, which enables one to write λ_3 explicitly as a simple function of λ_1 and λ_2

$$\lambda_3 = -\Big(\frac{\lambda_1\lambda_2}{\frac{\Lambda}{2}\lambda_1\lambda_2 + \lambda_1 + \lambda_2}\Big). \tag{20}$$

We will ultimately regard the two eigenvalues λ_1 and λ_2 as corresponding to two physical degrees of freedom for general relativity in the new variables.

One can see that (12) and (20) constitute a general solution, for non-degenerate B^i_a, for the constraints of GR corresponding to invariance under spatial and timelike diffeomorphisms. Equations (12) and (20) can be seen as the direct analogues of the solutions to (2) and (3), which would suffice to construct a solution if not for the Gauss' law constraint. We now argue that the Gauss' law constraint is superfluous as follows. Under the CDJ Ansatz, (7) reduces to

$$G_a[A^a_0] = \int_\Sigma d^3x \mathbf{w}_e\{\Psi_{ae}\}, \tag{21}$$

where we have defined the 'twisted' vector field $\mathbf{w}_e$ by

$$\mathbf{w}_e\{\Psi_{ae}\} = \mathbf{v}_e\{\Psi_{ae}\} + \big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big)A^b_i B^i_e \Psi_{fg} \tag{22}$$

with $\mathbf{v}_a = B^i_a \partial_i$. The unsmeared form of the Gauss' law constraint is given by $\mathbf{w}_e\{\Psi_{ae}\} = 0$, which in the parametrization (9) yields

$$\epsilon_{aed}\mathbf{w}_e\{\psi_d\} + \mathbf{w}_e\{\lambda_f O_{fa} O_{fe}\} = 0. \tag{23}$$

On the space of solutions to the diffeomorphism constraint $\psi_d = 0$, equation (23) reduces to

$$\mathbf{w}_e\{\lambda_f (e^{\vec{\theta}\cdot T})_{fa} (e^{\vec{\theta}\cdot T})_{fe}\} = 0. \tag{24}$$

Equation (24) is a set of three simultaneous first order partial differential equations in six unknowns λ_f and θ^f for $f = 1,2,3$. For each $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, one should in principle be able to find $\vec{\theta} = \vec{\theta}[\vec{\lambda}; B^a_i; \vec{\beta}]$, where $\vec{\beta}$ are the boundary data for $\vec{\theta}$.[4] The point is that while the Gauss' law constraint constitutes a genuine reduction in degrees of freedom of the CDJ

[4]The solution to (24) is the topic of a separate paper, which we do not display here.

matrix Ψ_{ae}, it is superfluous when one restricts oneself to the space of eigenvalues λ_f, since these eigenvalues subject to (20) are freely specifiable. Therefore equation (20) does in fact constitute a reduction of GR to two unconstrained degrees of freedom, the physical interpretation of which according to (24) fixes a unique $SO(3,C)$ frame $\vec{\theta}[\vec{\lambda};B^a_i;\vec{\beta}]$ by a nonlinear Lorentz transformation. The implication is that on the gauge-invariant, diffeomorphism-invariant subspace of GR, one still has the freedom to choose three free functions $\lambda_1(x)$, $\lambda_2(x)$ and $\lambda_3(x)$. We will exploit this observation to describe the dynamics of GR as a constrained system containing only one constraint, namely the Hamiltonian constraint, and devote the remainder of the paper to performing a consistency check on this dynamics.

3 Classical equations of motion

Usually in a theory of first class constrained systems one starts from a phase space ω posessing a cotangent bundle structure and then obtains the reduced phase space ω_{red} upon solving the constraints and factoring out the gauge orbits. Our approach might seem counterintuitive in that we perform the steps in reverse, namely by first imposing the cotangent bundle structure on the unconstrained degrees of freedom, while retaining the option to enlarge the phase space the constrained version of full general relativity by adding in the unphysical degrees of freedom by hand.

Associate to the eigenvalues of the CDJ matrix Ψ_{ae}, a set of momentum space dynamical variables $\Psi_f\equiv(\Psi_1,\Psi_2,\Psi_3)$ and define a phase space $\boldsymbol{\omega}\equiv(X^f,\Psi_f)$, where $X^f\equiv(X^1,X^2,X^3)$ are the set variables on the configuration space Γ canonically conjugate to Ψ_f. The effect is to impose the symplectic structure

$$\Omega=(\hbar G)^{-1}\int_\Sigma d^3x\delta\Psi_f(x)\wedge\delta X^f(x) \tag{25}$$

on the theory at the level prior to imposition of the Hamiltonian constraint, with the kinematic constraints having already being taken into account. The symplectic two form Ω implies the following elementary Poisson bracket

$$\{X^f(x),\Psi_f(y)\}=iG\delta^f_g\delta^{(3)}(\mathbf{x},\mathbf{y}) \tag{26}$$

for the basic variables. We now proceed to the dynamics of the theory, by considering its Hamiltonian evolution at the level of the gauge-invariant, diffeomorphism invariant subspace of full GR. Starting from a theory with Hamiltonian density

$$H[N]=\int_\Sigma d^3xN(\text{det}B)^{1/2}(\Psi_1\Psi_2\Psi_3)^{1/2}\Big(\frac{\Lambda}{2}+\frac{1}{\Psi_1}+\frac{1}{\Psi_2}+\frac{1}{\Psi_3}\Big) \tag{27}$$

we will need to obtain the Hamilton's equations of motion, which entails the calculation of variational derivatives of (27) with respect to the phase space variables (X^f,Ψ_f). The variational derivative with respect to the momentum Ψ_f is given by

$$\begin{aligned}\frac{\delta H[N]}{\delta\Psi_f(x)}&=\int_\Sigma d^3yN(y)(\text{det}B(y))^{1/2}\Big[\Big(\frac{\delta(\text{det}\Psi(y))^{1/2}}{\delta\Psi_f(x)}\Big)\Big(\frac{\Lambda}{2}+\text{tr}\Psi^{-1}(y)\Big)\\&\qquad+(\text{det}\Psi(y))^{1/2}\frac{\delta}{\delta\Psi_f(x)}\text{tr}\Psi^{-1}(y)\Big]\\&=\int_\Sigma d^3yN(y)(\text{det}B(y))^{1/2}\Big[\frac{1}{2}(1/\Psi_f(y))(\text{det}\Psi(y))^{1/2}\Big(\frac{\Lambda}{2}+\text{tr}\Psi^{-1}(y)\Big)\\&\qquad-(\text{det}\Psi(y))^{1/2}(1/\Psi_f(y))^2\Big]\delta^{(3)}(\mathbf{x},\mathbf{y}).\end{aligned} \tag{28}$$

The quantities $(1/\Psi_f(x))$ are actually the reciprocals of the respective eigenvalues Ψ_f, and therefore are defined only for $\Psi_f\neq 0$ and likewise for the reciprocals of the squares. Upon integration of the delta function, we have

$$\frac{\delta H[N]}{\delta\Psi_f(x)}=N(\text{det}B)^{1/2}\Big[-\sqrt{\text{det}\Psi}(\Psi_f)^{-2}+\frac{1}{2}(\Psi_f)^{-1}\sqrt{\text{det}\Psi}\Big(\frac{\Lambda}{2}+\text{tr}\Psi^{-1}\Big)\Big] \tag{29}$$

where we have used the Liebniz rule in conjuction with the nondegeneracy of the CDJ matrix Ψ_{ae}.

Moving on to the variational derivatives with respect to the configuration variable X^f we have

$$\frac{\delta H[N]}{\delta X^f(x)}=\int_\Sigma d^3y\Big(\frac{\delta(\text{det}B(y))^{1/2}}{\delta X^f(x)}\Big)N(y)(\text{det}\Psi(y))^{1/2}\Big(\frac{\Lambda}{2}+\text{tr}\Psi^{-1}(y)\Big). \tag{30}$$

Assuming $\text{det}B\neq 0$ we can include a factor of 1 in the form $(\text{det}B)^{-1/2}(\text{det}B)^{1/2}$, which allows us to extract the Hamiltonian constraint from the last factor of (30). The result is

$$\begin{aligned}\frac{\delta H[N]}{\delta X^f(x)}&=\int_\Sigma d^3y\Big(\frac{\delta(\text{det}B(y))^{1/2}}{\delta X^f(x)}\Big)(\text{det}B)^{-1/2}N(y)H(y)\\&=\frac{1}{2}\int_\Sigma d^3y(B^{-1}(y))^a_i\frac{\partial B^i_a(y)}{\partial X^f(x)}N(y)H(y)\delta^{(3)}(\mathbf{x},\mathbf{y}).\end{aligned} \tag{31}$$

The variational derivative with respect to the configuration variables X^f on configurations of nondegenerate Ashtekar magnetic fields B^i_a reduces to

$$\frac{\delta H[N]}{\delta X^f(x)} = \frac{1}{2}(B^{-1})^i_a\Big(\frac{\partial B^i_a}{\partial X^f}\Big)N\sqrt{\mathrm{det}B}\sqrt{\mathrm{det}\Psi}\Big(\frac{\Lambda}{2}+\mathrm{tr}\Psi^{-1}\Big) \tag{32}$$

upon integration of the delta function in (31). Immediately enter quantities B^i_a which have not been defined on the starting phase space (X^f, Ψ_f). In order to explicitly evalute (32) we would need to find $\partial B^i_a/\partial X^f$, which requires one to be able to express X^f explicitly in terms of the Ashtekar magnetic field B^i_a.[5] Whether it is possible to do this or not in the full theory for our purposes is immaterial, since on configurations of nondegenerate B^i_a, (32) is directly proportional to the Hamiltonian constraint which we will require to vanish.[6]

From (29) and (32) we can write the Hamilton's equations of motion on the gauge-invariant, diffeomorphism-invariant subspace of general relativity. These are given by

$$\dot{X}^f = \frac{\delta H[N]}{\delta \Psi_f} = N\Big(\frac{1}{2}(\Psi_f)^{-1}H - (\mathrm{det}B)^{1/2}(\mathrm{det}\Psi)^{1/2}(\Psi_f)^{-2}\Big) = 0 \tag{33}$$

and

$$\dot{\Psi}_f = -\frac{\delta H[N]}{\delta X^f} = \frac{1}{2}(B^{-1})^i_a\Big(\frac{\partial B^i_a}{\partial X^f}\Big)NH = 0, \tag{34}$$

which can be seen as having arisen from the first order starting action

$$I = \int_\Sigma d^3x\Big[\Psi_f\dot{X}^f - N\sqrt{\mathrm{det}B}\sqrt{\mathrm{det}\Psi}\Big(\frac{\Lambda}{2}+\mathrm{tr}\Psi^{-1}\Big)\Big]. \tag{35}$$

The evolution of the system must be confined to the constraint surface $H=0$ as a requirement of consistency of the system. Therefore, all terms proportional to the Hamiltonian constraint must vanish and the equations of motion reduce to

$$\dot{X}^f = -N(\mathrm{det}B)^{1/2}(\mathrm{det}\Psi)^{1/2}(\Psi_f)^{-2};\quad \dot{\Psi}_f = 0. \tag{36}$$

[5]Since the transformation from Ashtekar variables into the new variables $(A^a_i, \widetilde{\sigma}^i_a) \rightarrow (X^f, \Psi_f)$ is noncanonical, then it may be nontrivial to do this in closed form in the full theory.

[6]As part of the requirements of a consistent theory of constrained systems, the constraints surface must be preserved by Hamiltonian evolution and vice versa.

Note that when the Hamiltonian constraint H is satisfied, that $\dot{\Psi}_f = 0$, which means that Ψ_f can at most be an arbitrary function of spatial position $\mathbf{x} \in \Sigma$. Hence we have that $\Psi_f = \lambda_f$ are independent of time, and the equations of motion for X^f reduce to

$$\dot{X}^f = -N(\text{det}B)^{1/2}(\text{det}\lambda)^{1/2}(\lambda_f)^{-2}. \tag{37}$$

The time evolution of X^f is completely determined by the time evolution of $\text{det}B$, which naively appears to be unspecified based on the starting action (35) since the Ashtekar magnetic field B^i_a is not explictly part of the canonical structure (25).

3.1 Algebra of constraints

Some additional variables of this type will include various quantities related to the Ashtekar connection

$$\delta X^{ae} = B^i_e \delta A^a_i; \quad \delta X^{cg} = O^{cg}_f \delta X^f. \tag{38}$$

Suffice it for now to use these definitions (38) to make the result of (30) more readable. Continuing with (30), we have

$$\frac{\delta H[N]}{\delta X^f(x)} = \frac{1}{2}\int_\Sigma d^3y (B^{-1}(y))^a_i \int_\Sigma d^3z \int_\Sigma d^3z'$$
$$\Big(\frac{\delta X^{cg}(z)}{\delta X^f(x)}\Big)\Big(\frac{\delta A^b_j(z')}{\delta X^{cg}(z)}\Big)\Big(\frac{\delta B^i_a(y)}{\delta A^b_j(z')}\Big) N(y)H(y)$$
$$= \frac{1}{2} D^{ij}_{ab}\Big((B^{-1})^a_i (B^{-1})^g_j O^{bg}_f NH\Big). \tag{39}$$

The derivative in D^{ij}_{ab} acts on all terms in brackets, and arises from an integration by parts.

We now compute the Poisson bracket of two Hamiltonian constraints using

$$\{H[N], H[M]\} = \int_\Sigma d^3x \Big(\frac{\delta H[N]}{\delta \Psi_f(x)} \frac{\delta H[M]}{\delta X^f(x)} - \frac{\delta H[M]}{\delta \Psi_f(x)} \frac{\delta H[N]}{\delta X^f(x)}\Big)$$
$$= \int_\Sigma d^3x \Big[\Big(\frac{1}{2}(\Psi_f)^{-1} NH - N(\text{det}B)^{1/2}(\text{det}\Psi)^{1/2}(\Psi_f)^{-2}\Big)$$
$$\Big(\frac{1}{2} D^{ij}_{ab}((B^{-1})^a_i (B^{-1})^g_j O^{bg}_f MH\Big) - M \leftrightarrow N\Big] \tag{40}$$

(39). As one normally expects from the Hamiltonian constraint, which is a scalar, only the terms with derivatives acting on the lapses M and N survive, yielding

$$\{H[N],H[M]\}=\Pi^k[N\partial_k M-M\partial_k N], \tag{41}$$

where we have defined Π_k by

$$\Pi_k=\frac{1}{2}\epsilon^{ijk}(B^{-1})^a_i(B^{-1})^g_j O^{ag}_f\Big(\frac{1}{2}(\Psi_f)^{-1}H-(\text{det}B)^{1/2}(\text{det}\Psi)^{1/2}(\Psi_f)^{-2}\Big)H. \tag{42}$$

The Poisson of two Hamiltonian constraints is proportional to a Hamiltonian constraint with structure functions, which implies a closure of the algebra and therefore Dirac consistency. Therefore on the space of solutions to the Gauss' law and diffeomoeprhism constraints, the dynamics of the Hamiltonian is self-consistent.[7] The conclusion is that in the physical Soo variables, the algebra of the Hamiltonian constraint by itself forms a first class constrained system [**?**].[8] If one were to rather attempt to interpret (41) as a diffeomorphism, then it would imply a nonclosure of the algebra.

3.2 A toy model

It is obvious that on the configruation space $X^f\in\Gamma$, the magnetic field B^i_a contains seven unphysical degrees of freedom. While this may be the case, the possibility exists that the physical degrees of freedom inherent in B^i_a which are of relevance to the new variables X^f may reside entirely within $\text{det}B$.[9] This notion can be motivated by consideration of the spatially homogeneous sector of configuration space Γ as a toy model.[10] In this case the Ashtekar magnetic field is given by

$$B^i_a=\epsilon^{ijk}\partial_j A^a_k+\frac{1}{2}\epsilon^{ijk}f_{abc}A^b_jA^c_k=(\text{det}A)(A^{-1})^i_a. \tag{43}$$

[7]The algebra is still an 'open algebra' in the sense that there exist structure functions dependent on the phase space variables.

[8]In the original Ashtekar variables, the algebra of the Hamiltonian constraint yields a diffeomorphism constant and therefore does not close. From the vantage point of the Soo variables, the Hamiltonian constraint would form its own subalgebra, with the diffeomorphisms constituting a set of second class constraints.

[9]The relation $(X^1,X^2,X^3)\sim(b_1,b_2,b_3)$, where b_a are the eigenvalues of the Ashtekar magnetic field B^i_a seems appealing since this would lead to the relation $\text{det}B=b_1b_2b_3$. A rigorous proof of this is beyond the scope of the present paper, therefore we will for the time being regard it as a conjecture.

[10]We must emphasize that the homogeneous sector of Γ does not necessarily mean that we are considering minisuperspace. Even if the configuration space variables are spatially homogeneous, this dones not necessarily imply that the spatial homogeneity automatically extends to the momentum space variables Ψ_f.

In (43) we have used the fact that the structure constants $f_{abc} \equiv \epsilon_{abc}$ are numerically the same as the epsilon symbol ϵ_{ijk}, a property unique to three dimensional space Σ. While X^f may not have a simple expression in terms of the Ashtekar variables, the quantity $T = X^1 + X^2 + X^3$ integrated over Σ yields

$$\int_\Sigma d^3x T(x) = \int_\Sigma \int_\Gamma B^i_a(x)\delta A^a_i(x) = \int_\Sigma \mathrm{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big) = I_{CS}[A] \quad (44)$$

which is the Chern–Simons functional of the Ashtekar connection A^a_i. In the spatially homogeneous sector of configuration space, we have from (43) that $\det B = (\det A)^2$, and also from the integrand of (44) that $T = \det A$. Summing (37) over the index f we have

$$\dot{T} = \sum_{f=1}^3 \dot{X}^f = -N\eta_\lambda T, \tag{45}$$

where we have defined

$$\begin{aligned} \eta_\lambda &= \frac{1}{2}\sqrt{\lambda_1\lambda_2\lambda_3}\Big[\Big(\frac{1}{\lambda_1}\Big)^2 + \Big(\frac{1}{\lambda_2}\Big)^2 + \Big(\frac{1}{\lambda_3}\Big)^2\Big] \\ &= \frac{1}{2}\lambda_1\lambda_2\left(\frac{\frac{\Lambda^2}{2} + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_1\lambda_2} + \frac{\Lambda}{\lambda_1} + \frac{\Lambda}{\lambda_2}}{\frac{\Lambda}{2}\lambda_1\lambda_1 + \lambda_1 + \lambda_2}\right) \end{aligned} \tag{46}$$

using (20). One can immediately detemine the evolution of $\det B$ in time, which fixes the time evolution of the combination $T = X^1 + X^2 + X^3$, by integrating (45) to yield

$$T(t) = T_0 e^{-\eta_\lambda \int_0^t N(t)} \tag{47}$$

Note from (47) that there is still freedom in the choice of the lapse function N, which determines the manner of evolution normal to spatial hypersurfaces Σ. For numerically constant N this leads to exponentially inflating solutions.

4 Quantization of the unconstrained phase space

Upon quantization we must first promote the dynamical variables X^f and Ψ_f to quantum operators $\hat{X}^f$ and $\hat{\Psi}_f$ and the Poisson brackets (26) to equal-time commutation relations

$$\big[\hat{X}^f(\mathbf{x},t),\hat{\Psi}_g(\mathbf{y},t)\big]=G\delta^f_g\delta^{(3)}(\mathbf{x},\mathbf{y}). \tag{48}$$

In the functional Schrödinger representation, the quantum wavefunction is a functional of the configuration variables $\boldsymbol{\psi}=\boldsymbol{\psi}[X]$ and the basic operators act respectively by multiplication and by functional differentiation

$$\hat{X}^f(\mathbf{x},t)\boldsymbol{\psi}[X]=(X^f(\mathbf{x},t))\boldsymbol{\psi}[X];$$
$$\hat{\Psi}_f(\mathbf{x},t)\boldsymbol{\psi}[X]=(\hbar G)\frac{\delta}{\delta X^f(\mathbf{x},t)}\boldsymbol{\psi}[X]. \tag{49}$$

4.1 Normalizability of the quantum wavefunctional

There are two main possibilities for Hilbert spaces in the Soo–CDJ variables, depending on the signature of spacetime. For spacetimes of Euclidean signature, all variables can be regarded as real. Hence the measure in this case can be given by

$$D\mu_{Eucl}(X)=DX\equiv\prod_{f=1}^3\bigotimes_{\mathbf{x}}\nu^{-1}\delta X^f(\mathbf{x}) \tag{50}$$

Upon the rescaling $\lambda_f\rightarrow i\lambda_f$ the wavefunctions, which are now a pure phase, become delta-functional normalizable

$$\begin{aligned}\big\langle\boldsymbol{\psi}_{\vec{\lambda}}\big|\boldsymbol{\psi}_{\vec{\zeta}}\big\rangle_{Eucl}&=\prod_{\mathbf{x},f}\int_\Gamma\nu^{\zeta(0)}\delta X^f\\ \exp\Big[i(\hbar G)^{-1}\int_\Sigma d^3x\lambda_f(x)\overline{X}^f(x)\Big]&\exp\Big[-i(\hbar G)^{-1}\int_\Sigma d^3x\zeta_f(x)Xf(x)\Big]\\ &=\prod_x\delta\big(\lambda_f(x)-\zeta_f(x)\big).\end{aligned} \tag{51}$$

The interpretation is that spacetimes of Euclidean signature not globally having the same algebraic classification are orthogonal.

For spacetimes of Lorentzian signature, it is not possible to impose orthogonality by delta functions using the Lebesgue measure (50). Since we require square integrable wavefunctions as a necessary condition for a sensible quantum theory, we must use a measure which falls off faster than the wavefunctions can blow up as $X\rightarrow\infty$. Hence, we define the physical Hilbert space for Lorentzian signature spacetimes as the set of square integrable functions $\boldsymbol{\Psi}_\lambda[X]\in\boldsymbol{H}_{Phys}$ with respect to the Gaussian measure

$$
D\mu_{Lor}(X) = \bigotimes_{\mathbf{x}} \nu^{-1}\delta X e^{-\nu^{-1}\overline{X}\cdot X}
$$

$$
= \prod_{\mathbf{x},f} \delta X^f \exp\Big[-\nu^{-1}\int_\Sigma d^3x \overline{X}^f(x) X_f(x)\Big]. \qquad (52)
$$

The inner product of two physical states for Lorentzian signature is now given by

$$
\langle \boldsymbol{\psi}_{\vec{\lambda}} | \boldsymbol{\psi}_{\vec{\zeta}} \rangle_{Lor} = \prod_{\mathbf{x}f} \int_\Gamma \nu^{\zeta(0)} \delta X^f \exp\Big[-\nu^{-1}\int_\Sigma d^3x \overline{X}^f(x) X_f(x)\Big]
$$

$$
\exp\Big[(\hbar G)^{-1}\int_\Sigma d^3x \lambda^*_f(x) \overline{X}^f(x)\Big] \exp\Big[(\hbar G)^{-1}\int_\Sigma d^3x \zeta_f(x) X f(x)\Big]
$$

$$
= \exp\Big[\nu(\hbar G)^{-2}\int_\Sigma d^3x \lambda^*_f(x)\zeta_f(x)\Big]. \qquad (53)
$$

The requirement of normalizability is equivalent to square integrability of $\lambda_f(x)$ and $\zeta_f(x)$. On a final note, we point out that the pure Kodama state Ψ_{Kod} exists as a member of the physical Hilbert space $\boldsymbol{\psi}_{\vec{\lambda}}$, namely for the choice $\lambda_1 = \lambda_2 = \lambda_3 = -\frac{6}{\Lambda}$. In this case, the state would label a spacetime of Petrov type O.

5 Quantum Hamiltonian constraint: revisited

There are two main considerations when quantizing the Hamiltonian constraint, namely operator-ordering and regularization. Operator ordering is not an issue in our case since all momenta occur to the right upon quantization, which leaves regularization of products of momenta. Let us define the quantized version of the Hamiltonian constraint as the result of averaging the constraint with all of its transposes, so that it is Hermitian. Hence, upon quantization we have that

$$
\hat{H} = I_{fg}\hat{\Psi}_f\hat{\Psi}_g + \Lambda I_{fgh}\hat{\Psi}_f\hat{\Psi}_g\hat{\Psi}_h
$$

$$
= \big(\hat{\Psi}_1\hat{\Psi}_2 + \hat{\Psi}_2\hat{\Psi}_1 + \hat{\Psi}_2\hat{\Psi}_3 + \hat{\Psi}_3\hat{\Psi}_2 + \hat{\Psi}_3\hat{\Psi}_1 + \hat{\Psi}_1\hat{\Psi}_3\big)
$$

$$
+\Lambda\Big(\hat{\Psi}_1\hat{\Psi}_2\hat{\Psi}_3 + \hat{\Psi}_2\hat{\Psi}_3\hat{\Psi}_1 + \hat{\Psi}_3\hat{\Psi}_1\hat{\Psi}_2 + \hat{\Psi}_2\hat{\Psi}_1\hat{\Psi}_3 + \hat{\Psi}_1\hat{\Psi}_3\hat{\Psi}_2 + \hat{\Psi}_3\hat{\Psi}_2\hat{\Psi}_1\Big), \qquad (54)
$$

where I_{fg} and I_{fgh} are tensors which impose the desired symmetry averaging upon contraction with the appropriate factors of Ψ_f.

To regularize the Hamiltonian constraint we make use of the results of [**?**], which uses a point-splitting regularization procedure combined with

smearing of each individual factor in the operator product. We make use of a regulating function $f_\epsilon(\mathbf{x},\mathbf{y})$, which is a continuous function of a regularizing parameter ϵ, such that

$$\int_\Sigma d^3x f_\epsilon(\mathbf{x},\mathbf{y})\varphi(y)=\varphi(x) \tag{55}$$

for all smooth test functions $\varphi\in C^\infty(\Sigma)$. The ingredients of the quantized Hamiltonian constraint are given by

$$\hat{H}_{ae}(x)=\hat{\Psi}_a(x)\hat{\Psi}_e(x);\quad \hat{H}_{fae}(x)=\hat{\Psi}_f(x)\hat{\Psi}_a(x)\hat{\Psi}_e(x). \tag{56}$$

In order to obtain a well-defined action of the constraint of the wavefunctional, it is not necessary to regularize the first operator since it already produces a finite result upon action on the wavefunction. To regularize we start with the quadratic term, going into the Schrödinger representation

$$\hat{H}^\epsilon_{ae}\boldsymbol{\psi}=(\hbar G)^2\int_\Sigma d^3y f_\epsilon(\mathbf{x},\mathbf{y})\frac{\delta^2}{\delta X^a(y)\delta X^e(x)}\boldsymbol{\psi}[X]. \tag{57}$$

Using the general form of the wavefunction as a third-quantized Wilson line

$$\boldsymbol{\psi}=\exp\Big[\int_\Sigma d^3x\int_\Gamma \lambda_f[X(x)]\delta X^f(x)\Big], \tag{58}$$

the first functional derivative brings down a factor of λ_f as in

$$(\hbar G)\frac{\delta}{\delta X^f(x)}\boldsymbol{\psi}[X]=\lambda_f[X(x)]\boldsymbol{\psi}[X]. \tag{59}$$

As previously mentioned, it is not necessary to regularize this first factor (59). Continuing from (57) and regularizing the second factor, we have

$$\begin{aligned}&(\hbar G)\int_\Sigma d^3y f_\epsilon(\mathbf{x},\mathbf{y})\frac{\delta}{\delta X^a(y)}(\lambda_e(x)\boldsymbol{\psi})\\
&=\int_\Sigma d^3y f_\epsilon(\mathbf{x},\mathbf{y})\Big[\hbar G\Big(\frac{\partial\lambda_e}{\partial X^a}\Big)_x\delta^{(3)}(\mathbf{x},\mathbf{y})+\lambda_e(x)\lambda_a(y)\Big]\boldsymbol{\psi}\\
&=\Big((\hbar G f_\epsilon(0))\Big(\frac{\partial\lambda_e}{\partial X^a}\Big)_x+\int_\Sigma d^3y f_\epsilon(\mathbf{x},\mathbf{y})\lambda_a(x)\lambda_e(y)\Big)\boldsymbol{\psi}.\end{aligned} \tag{60}$$

where we have defined $f_\epsilon(0)\equiv f_\epsilon(\mathbf{x},\mathbf{x})$ and we have used (59).

Moving on to the cubic term, we have

$$\hat{H}^{\epsilon}_{fae}\boldsymbol{\psi} = \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{y})) \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \hat{\Psi}_f(z) \hat{\Psi}_a(y) \hat{\Psi}_e(x) \boldsymbol{\psi}$$

$$= (\hbar G)^3 \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{y}) \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \frac{\delta^3}{\delta X^f(z) \delta X^a(y) \delta X^e(x)} \boldsymbol{\psi}. \tag{61}$$

The first two functional derivatives in (61) have already been evaluated in (60). Hence, continuing from (60), we have

$$\hat{H}^{\epsilon}_{fae}\boldsymbol{\psi} = (\hbar G) \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z}) \frac{\delta}{\delta X^f(z)}$$

$$\Big((\hbar G f_{\epsilon}(0)) \Big(\frac{\partial \lambda_e}{\partial X^a} \Big)_x + \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \lambda_a(x) \lambda_e(y) \Big) \boldsymbol{\psi}$$

$$= \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z}) \Bigg[(\hbar G)^2 f_{\epsilon}(0) \Big(\frac{\partial^2 \lambda_e}{\partial X^f \partial X^a} \Big)_x \delta^{(3)}(\mathbf{x},\mathbf{z})$$

$$+ (\hbar G) \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \Big(\lambda_a(x) \Big(\frac{\partial \lambda_e}{\partial X^f} \Big)_y \delta^{(3)}(\mathbf{z},\mathbf{y}) + \lambda_e(y) \Big(\frac{\partial \lambda_a}{\partial X^f} \Big)_x \delta^{(3)}(\mathbf{x},\mathbf{z})$$

$$+ (\hbar G) f_{\epsilon}(0) \lambda_f(x) \Big(\frac{\partial \lambda_e}{\partial X^a} \Big)_x + \lambda_f(z) \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \lambda_a(x) \lambda_e(y) \Bigg] \boldsymbol{\psi}. \tag{62}$$

We now evalute each individual term of (62). The term of order $(\hbar G)^2$ is given by

$$(\hbar G)^2 f_{\epsilon}(0) \int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z}) \Big(\frac{\partial^2 \lambda_e}{\partial X^f \partial X^a} \Big)_x \delta^{(3)}(\mathbf{x},\mathbf{z}) = (\hbar G f_{\epsilon}(0))^2 \Big(\frac{\partial^2 \lambda_e}{\partial X^f \partial X^a} \Big)_x . \tag{63}$$

The term of order $\hbar G$ has three contributions. The first contribution is given by

$$(\hbar G \Big[\int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z}) \int_{\Sigma} f_{\epsilon}(\mathbf{x},\mathbf{y}) \lambda_a(x) \Big(\frac{\partial \lambda_e}{\partial X^f} \Big)_y \delta^{(3)}(\mathbf{z},\mathbf{y}) \Big]$$

$$= (\hbar G) \lambda_a(x) \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) f_{\epsilon}(\mathbf{x},\mathbf{y}) \Big(\frac{\partial \lambda_e}{\partial X^f} \Big)_y \equiv (\hbar G) \alpha^{\epsilon}_{aef}(x) \tag{64}$$

where we have integrated d^3z to eliminate the delta function. The second contribution of order $\hbar G$ is given by

$$(\hbar G) \Big[\int_{\Sigma} d^3z f_{\epsilon}(\mathbf{x},\mathbf{z}) \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \lambda_e(y) \Big(\frac{\partial \lambda_a}{\partial X^f} \Big)_x \delta^{(3)}(\mathbf{x},\mathbf{z}) \Big]$$

$$= (\hbar G f_{\epsilon}(0)) \Big(\frac{\partial \lambda_a}{\partial X^f} \Big)_x \int_{\Sigma} d^3y f_{\epsilon}(\mathbf{x},\mathbf{y}) \lambda_e(y) \equiv (\hbar G f_{\epsilon}(0)) \beta^{\epsilon}_{aef}(x) \tag{65}$$

where we have integrated d^3z to get rid of the delta function, in conjunction with factorization of the x dependence. The third contribution of order $\hbar G$ is given by

$$(\hbar G)\Big[\int_\Sigma d^3z f_\epsilon(\mathbf{x},\mathbf{z})f_\epsilon(0)\lambda_f(z)\Big(\frac{\partial\lambda_e}{\partial X^a}\Big)_x\Big]$$
$$= \hbar G f_\epsilon(0)\Big(\frac{\partial\lambda_e}{\partial X^a}\Big)_x\int_\Sigma d^3z f_\epsilon(\mathbf{x},\mathbf{z})\lambda_f(z) = (\hbar G f_\epsilon(0))\gamma^\epsilon_{aef}(x). \tag{66}$$

The semiclassical term is given by

$$S^\epsilon_{aef}(x) = \lambda_a(x)\Big(\int_\Sigma d^3z f_\epsilon(\mathbf{x},\mathbf{z})\lambda_f(z)\Big)\Big(\int_\Sigma d^3z f_\epsilon(\mathbf{x},\mathbf{y})\lambda_e(y)\Big). \tag{67}$$

The next step of the regularization procedure is to isolate the singularities in the constraint to poles in ϵ, in conjunction with taking the $\epsilon\rightarrow 0$ limit. To this end we will make use of a technique widely used in field theory, namely that it is safe to replace quantities with their limits as long as they do not blow up. We will repeatedly make use of (55) in the $\epsilon\rightarrow 0$ limit for all quantities occuring under an integral sign, isolating any poles to the orders of singularity. This is necessary only for the terms of order $\hbar G$.

$$\text{lim}_{\epsilon\rightarrow 0}\alpha^\epsilon_{aef}(x) = \lambda_a(x)\text{lim}_{\epsilon\rightarrow 0}\int_\Sigma d^3y f_\epsilon(\mathbf{x},\mathbf{y})\Big(f_\epsilon(\mathbf{x},\mathbf{y})\Big(\frac{\partial\lambda_a}{\partial X^f}\Big)_y\Big) = f_\epsilon(0)\lambda_a(x)\Big(\frac{\partial\lambda_a}{\partial X^f}\Big)_x \tag{68}$$

where we have identified the term in (68) in brackets with $\varphi\in C^\infty(\Sigma)$. Next, we have

$$\text{lim}_{\epsilon\rightarrow 0}\beta^\epsilon_{aef}(x) = \Big(\frac{\partial\lambda_a}{\partial X^f}\Big)_x\text{lim}_{\epsilon\rightarrow 0}\int_\Sigma d^3y f_\epsilon(\mathbf{x},\mathbf{y})\lambda_3(y) = \Big(\frac{\partial\lambda_a}{\partial X^f}\Big)_x\lambda_e(x) \tag{69}$$

where we have identified $\varphi(y)\equiv\lambda_e(y)$. Next, we have

$$\text{lim}_{\epsilon\rightarrow 0}\gamma^\epsilon_{aef}(x) = \Big(\frac{\partial\lambda_a}{\partial X^e}\Big)_x\text{lim}_{\epsilon\rightarrow 0}\int_\Sigma d^3y f_\epsilon(\mathbf{x},\mathbf{z})\lambda_f(z) = \Big(\frac{\partial\lambda_a}{\partial X^e}\Big)_x\lambda_f(x) \tag{70}$$

where we have identified $\varphi(z)\equiv\lambda_f(z)$. For the semiclassical term we have

$$\text{lim}_{\epsilon\rightarrow 0}S^\epsilon_{aef}(x) = \lambda_f(x)\lambda_a(x)\lambda_e(x) \tag{71}$$

The coefficients of the factors of $\hbar G$ and $(\hbar G)^2$ are finite, which enables the isolation of the poles to powers of $f_\epsilon(0)$.

Collecting all of the contributions, we contract the ingredients of the quantized Hamiltonian constraint with symmetry tensors I_{ae} and I_{fae} which impose the required symmetry (54), obtaining a regularized Hamiltonian constraint of

$$\hat{H}^{\epsilon}\boldsymbol{\psi} = \big(I^{ae}\hat{H}^{\epsilon}_{ae} + \Lambda I^{fae}\hat{H}^{\epsilon}_{fae}\big)\boldsymbol{\psi} = \Bigg[I^{ae}\lambda_a\lambda_e + \Lambda I^{fae}\lambda_f\lambda_a\lambda_e$$

$$+(\hbar G f_{\epsilon}(0))\Big(I^{ae}\frac{\partial\lambda_e}{\partial X^a} + \Lambda I^{fae}\lambda_a\frac{\partial\lambda_e}{\partial X^f}\Big) + \Lambda(\hbar G f_{\epsilon}(0))^2 I^{fae}\Big(\frac{\partial^2\lambda_e}{\partial X^f\partial X^a}\Big)\Bigg]\boldsymbol{\psi}. \quad (72)$$

The action of the regularized Hamiltonian constraint in the small ϵ limit can be written in the form

$$\hat{H}^{\epsilon}(x)\boldsymbol{\psi} = \big(q_0(x) + (\hbar G f_{\epsilon}(0))q_1(x) + (\hbar G f_{\epsilon}(0))^2 q_2(x)\big)\boldsymbol{\psi}. \quad (73)$$

5.1 Physical Hilbert space

In the usual methods of loop quantum gravity, one chooses a kinematical Hilbert space $\boldsymbol{H}_{kin}$ at the level prior to implementation of the constraints. In the Soo/CDJ variables one may choose wavefunctionals of the form

$$\boldsymbol{\psi}[X] = \otimes_{\mathbf{x}} e^{\int_{\Gamma}\lambda_f \delta X^f} = \exp\Big[\int_{\Sigma}\int_{\Gamma}\lambda_f[X]\delta X^f\Big], \quad (74)$$

where $\lambda_f[X]$ are at this stage arbitrary functions of X. Equation (74) already satisfies the diffeomorphism and the Gauss' law constraints, which leaves remaining the quantized Hamiltonian constraint. Therefore, we will need to apply (73) in the limit of removal of the regulator. Hence,

$$\hat{H}\boldsymbol{\psi} = \mathrm{lim}_{\epsilon\rightarrow 0}\hat{H}^{\epsilon}(x)\boldsymbol{\psi}$$
$$= \mathrm{lim}_{\epsilon\rightarrow 0}\big(q_0(x) + (\hbar G f_{\epsilon}(0))q_1(x) + (\hbar G f_{\epsilon}(0))^2 q_2(x)\big)\boldsymbol{\psi}. \quad (75)$$

Note that as $\epsilon \rightarrow 0$, we have that $f_{\epsilon}(0) \rightarrow \delta^{(3)}(0)$ and $(f_{\epsilon}(0))^2 \rightarrow (\delta^{(3)}(0))^2$. In fact, these delta functions of zero are precisely what one would obtain as the numerical coefficients of q_1 and q_2 if one were to formally compute the action of the Hamiltonian constraint unregularized on $\boldsymbol{\psi}$. Since the Hamiltonian constraint must be satisfied independently at each point, a sufficient condition is that $q_0 = q_1 = q_2 = 0$. We require that all wavefunctions must be independent of the specific form of the regulating function f_{ϵ} and must

be independent of any regulating parameters in the limit when the regulator is removed.[11] Since ϵ is arbitrary, it follows that $q_1 = q_2 = 0$ is necessary since these terms must vanish faster than any possible regulator can blow up, in order to satisfy the constraint. The result is a set of three equations which must be satisfied, namely

$$q_0 = I^{ae}\lambda_a\lambda_e + \Lambda I^{fae}\lambda_f\lambda_a\lambda_e = 0;$$
$$q_1 = I^{ae}\frac{\partial\lambda_e}{\partial X^a} + \Lambda I^{fae}\lambda_a\frac{\partial\lambda_e}{\partial X^f} = 0;$$
$$q_2 = \Big(\frac{\partial^2\lambda_e}{\partial X^f\partial X^a}\Big) = 0. \tag{76}$$

Equation (76) is a set of three equations in three unknowns $\lambda_1[X]$, $\lambda_2[X]$ and $\lambda_3[X]$, and we must construct a Hilbert space of states of the form (74) solving these equations. The general solution involves an integration in the functional space of fields Γ, which involves constants of functional integration with respect to X^f at each point at which the Hamiltonian constraint is satisfied.[12] We will be content, for the purposes of making contact with the previous sections, to consider wavefunctions for which λ_f is independent of the dynamical variable X^f. In this case the second and third equations of (76) automatically vanish, and we are left with the first equation which is the semiclassical part of the Hamiltonian constraint

$$2\big(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\big) + \Lambda\lambda_1\lambda_2\lambda_3 = 0 \ \forall x. \tag{77}$$

The general solution to (77) is given by

$$\lambda_3 = -\Big(\frac{\lambda_1\lambda_2}{\frac{\Lambda}{2}\lambda_1\lambda_2 + \lambda_1 + \lambda_2}\Big). \tag{78}$$

The corresponding wavefunctional, which then acquires the labels λ_1 and λ_2, is given by

$$\boldsymbol{\psi}_\lambda[X] = \exp\Big[(\hbar G)\int_\Sigma d^3x\lambda_f(x)X^f(x)\Big]. \tag{79}$$

Since the configuration variables X^f are now conjugate configuration variables for the eigenvalues of the symmetric part of the CDJ matrix Ψ_{ae}, then there is no dependence upon the variable conjugate to ψ_d. Hence we have

[11] This is the standard required of a finite state of quantum gravity by our interpretation.

[12] The integration over Γ therefore introduces continuous functions with each quadrature, which can be used to label the states.

that $\boldsymbol{\psi}_\lambda \in Ker\{\hat{H} \bigcap \hat{H}_i\}$, and whatever that conjugate variable is, it must be unphysical. Additionally, $\boldsymbol{\psi}_\lambda \in Ker\{\hat{G}_a\}$ since λ_1 and λ_2 fix the required $SO(3,C)$ frame for a given B^i_a.

$$\boldsymbol{\psi}_\lambda[X] = \exp\Bigl[\int_\Sigma d^3x \lambda_f(x) X^f(x)\Bigr]. \tag{80}$$

The functional (80), which satisfies the quantum Hamiltonian constraint, also satisfies the diffeomorphism constraint. Since the configuration variables X^f are now conjugate configuration variables for the eigenvalues of the symmetric part of the CDJ matrix Ψ_{ae}, then there is no dependence upon the variable conjugate to ψ_d. Hence we have that $\boldsymbol{\psi}_\lambda \in Ker\{\hat{H} \bigcap \hat{H}_i\}$, and whatever that conjugate variable is, it must be unphysical.

We will ultimately move on to the Gauss' law constraint, but let us first address the normalizability of the wavefunctions (80). Since in the worst case the variables X^f may be complex, we require $\boldsymbol{H}_{Kin} \subset L^2(\Gamma, D\mu_{Lor}(X))$ to be the set of square integrable functionals on the functional manifold Γ, with Gaussian measure

$$D\mu_{Lor}(X) = DXe^{-\overline{X}_f(x)X^f(x)} = \prod_{\mathbf{x}} \delta X(x) \exp\Bigl[-\int_\Sigma d^3x \overline{X}_f(x) X^f(x)\Bigr]. \tag{81}$$

6 Discussion: Hamilton–Jacobi functional

From (20) one may construct the Hamilton–Jacobi functional

$$S = \int_\Sigma d^3x \Bigl[\lambda_1 X^1 + \lambda_2 X^2 - \Bigl(\frac{\lambda_1\lambda_2}{\frac{\Lambda}{2}\lambda_1\lambda_2 + \lambda_1 + \lambda_2}\Bigr) X^3\Bigr], \tag{82}$$

which exponentiated in units of $\hbar G$ yields a wavefunctional

$$\boldsymbol{\psi}_\lambda[X] = e^{(\hbar G)^{-1}S}. \tag{83}$$

In (83) $\lambda_f \equiv (\lambda_1, \lambda_2, \lambda_3)$ are a set of labels and the variables $X^f \equiv (X^1, X^2, X^3)$ are a set of configuration space variables whose role will become clear.

To put (82) into a more physically intuitive form, let us rewrite it in terms of the trace and the traceless parts. First make the substitution $X^3 = T - X^1 - X^2$ into (82) to eliminate X^3 in favor of the trace T. The integrand of (82) reduces to the form $\alpha X + \beta Y - ET$, where we have defined

$X = X^1$ and $Y = X^2$. The following nonlinear relation ensues amongst α, β and λ_1 and λ_2

$$\beta = \lambda_2 + \frac{\lambda_1\lambda_2}{\frac{\Lambda}{2}\lambda_1\lambda_2 + \lambda_1 + \lambda_2} = \frac{(\frac{\Lambda}{2}\lambda_1 + 1)(\lambda_2)^2 + 2\lambda_1\lambda_2}{\frac{\Lambda}{2}\lambda_1\lambda_2 + \lambda_1 + \lambda_2} \tag{84}$$

and

$$\beta - \alpha = \lambda_2 - \lambda_1 \longrightarrow \lambda_2 = \lambda_1 + \beta = \alpha. \tag{85}$$

Substitution of (85) into (84) yields the following cubic polynomial equation for λ_1

$$(\frac{\Lambda}{2}\lambda_1 + 1)(\lambda_1 + \beta - \alpha)^2 + \big(2\lambda_1 - \beta(\frac{\Lambda}{2}\lambda_1 + 1)\big)(\lambda_1 + \beta - \alpha) - \beta\lambda_1 = 0, \tag{86}$$

which simplifies to

$$(\lambda_1)^3 + \Big(\frac{6}{\Lambda} + \beta - 2\alpha\Big)(\lambda_1)^2 - \Big(\frac{4\beta}{\Lambda} + \alpha(\beta - \alpha)\Big)\lambda_1 - \frac{2}{\Lambda}\alpha(\beta - \alpha) = 0 \tag{87}$$

To solve (87) using trigonometric functions, it helps to put it into dimensionless form using the rescaling $\lambda_1 = \frac{\rho}{\gamma}$, where γ is a numerical constant of dimension $[\gamma] = 2$ and ρ is dimensionless. Then (86) reduces to

$$\rho^3 + A\rho^2 + B\rho + C = 0, \tag{88}$$

where we have defined

$$A = \gamma^2\Big(\frac{6}{\Lambda} + \beta - 2\alpha\Big); \quad B = -\gamma^2\Big(\frac{4\beta}{\Lambda} + \alpha(\beta - \alpha)\Big); \quad C = -\gamma^3\Big(\frac{2}{\Lambda}\alpha(\beta - \alpha)\Big). \tag{89}$$

The general solution for the first eigenvalue λ_1 is given by

$$\lambda_1 = \lambda_1(\alpha, \beta) = -\frac{A}{3} + \sqrt{\frac{4}{3}\Big(B - \frac{A^3}{3}\Big)}T^r_{1/3}\left[\frac{-4(C - \frac{AB}{3} + \frac{2A^3}{27})}{\Big(\frac{4}{3}\Big(B - \frac{A^3}{3}\Big)\Big)^{3/2}}\right] \tag{90}$$

where $r = \frac{n\pi}{3}$ labels the the three roots of the cubic polynomial for $n = (0, \pm 1)$, and we have defined the Chebyshev polynomial

$$T^r_a(x) = \sin\big[a\sin^{-1}x + r\big]. \tag{91}$$

The solution for the second eigenvalue λ_2 automatically follows

$$\lambda_2 = \lambda_2(\alpha, \beta) = \lambda_1(\alpha, \beta) + \beta - \alpha. \tag{92}$$

Making the definition

$$E \equiv E(\alpha, \beta) = \frac{\lambda_1(\alpha, \beta)\lambda_2(\alpha, \beta)}{\frac{\Lambda}{2}\lambda_1(\alpha, \beta)\lambda_2(\alpha, \beta) + \lambda_1(\alpha, \beta) + \lambda_2(\alpha, \beta)}, \tag{93}$$

one can construct a wavefunction by exponentiating (82) of the form

$$\boldsymbol{\psi}_{\alpha\beta}[X, Y, T] = \exp\Big[-(\hbar G)^{-1}\int_\sigma d^3x\Big(\alpha(x)X(x) + \beta(x)Y(x) - E_{\alpha\beta}(x)T(x)\Big)\Big]. \tag{94}$$

Equation (94) is a gauge-invariant, diffeomorphism invariant wavefunctional which satsifies the classical Hamiltonian constraint by construction. As an aside, for $\alpha = \beta = 0$, we have that $E_{00} = \frac{6}{\Lambda}$ and (94) reduces to the pure Kodama state

$$\Psi_{Kod} = \boldsymbol{\psi}_{00}[X, Y, T] = e^{-6(\hbar G\Lambda)^{-1}\int_\Sigma T}. \tag{95}$$

Note that the trace $T = X^1 + X^2 + X^3$ is directly related via

$$T = \int_\Sigma\int_\Gamma B^i_a \delta A^a_i = \int_\Sigma \mathrm{tr}\Big(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\Big), \tag{96}$$

which is the Chern–Simons functional for the Ashtekar connection. The exponential of (96) yields the Kodama state Ψ_{Kod}, which is the both a semiclassical manifestation of DeSitter spacetime as well as a quantum state. For more general case (94), the semiclassical orbits encompass more general solutions to the Einstein's equations.

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